Scalar and Vector Potentials

How the sources (ρ and J) generate electric and magnetic fields → we seek the general sol^n to Maxwell's equations,

(i) $\nabla \cdot E = \frac{1}{\epsilon_0}\rho$ (iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

(ii) $\nabla \cdot B = 0$ (iv) $\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$ — (1)

Given $\rho(r,t)$ and $J(r,t)$ ⟶ $E(r,t)$ and $B(r,t)$?

In static case Coulomb's law and Biot–Savart law provide the answer.

Generalization of those laws to time-dependent configurations.

In electrostatics $\nabla \times E = 0 \longrightarrow E = -\nabla V$.

In electrodynamics ⟶ curl of E is nonzero

B → divergenceless

$$\Rightarrow \quad B = \nabla \times A \quad \text{— (2)}$$

as in magnetostatics

Putting this in Faraday's (iii) law

$$\nabla \times E = -\frac{\partial}{\partial t}(\nabla \times A)$$

or

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

Curl does vanish

$$E + \frac{\partial A}{\partial t} = -\nabla V$$

$$\Rightarrow \quad \boxed{E = -\nabla V - \frac{\partial A}{\partial t}} \quad \text{— (3)}$$

reduces to $E = -\nabla V$ if $A \to$ constant

Eq^n (2) and (3) $\to$ potential representation

$\downarrow$ fulfill automatically the two homogeneous Maxwell eq^{n}s, (ii) and (iii).

How about Gauss's law (i) and the Ampere/Maxwell law (iv)?

Putting (3) in (i)

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\rho/\varepsilon_0 \quad \text{---(4)}$$

This replaces Poisson's eq^n $\to$ static case.

Putting eq^n (2) & (3) in (iv)

$$\nabla \times (\nabla \times A) = \mu_0 J - \mu_0 \varepsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2}$$

or, using the vector identity $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$

$$\left(\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2}\right) - \nabla\left(\nabla \cdot A + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 J$$

$$\text{---(5)}$$

Gauge Transformations

eq^{n}s (4) & (5) $\to$ ugly

However, we have succeeded $\to$ in reducing six problems $\to$ finding E and B (three components each) — to

from: V (one component) and A (three component)

Eq^n (2) & (3) $\to$ do not uniquely define the potentials, we are free to impose extra conditions on V and A

$\downarrow$

as long as nothing happens to E and B.

Gauge freedom?

Suppose, we have two sets of potentials, (V, A) and (V', A'), which corresponds to the same electric and magnetic field.

By how much can they differ?

$A' = A + \alpha$ and $V' = V + \beta$

the A's give same B, their curls must be equal, and hence

$$\nabla \times \alpha = 0.$$

we can therefore write α as the gradient of some scalar

$$\alpha = \nabla \lambda$$

The two potentials also give same E, so

$$\nabla \beta + \frac{\partial \alpha}{\partial t} = 0$$

or $$\nabla \left(\beta + \frac{\partial \lambda}{\partial t} \right) = 0$$

independent of position

$\Rightarrow$ $\beta = -\frac{\partial \lambda}{\partial t} + k(t)$ However, depend on time

absorb $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')\,dt'$ to the old one.

This will not affect the gradient of λ; it just adds $k(t)$ to $\frac{\partial \lambda}{\partial t}$

$$\Rightarrow \quad \left.\begin{aligned} A' &= A + \nabla \lambda \\ V' &= V - \frac{\partial \lambda}{\partial t} \end{aligned}\right| \quad \text{--- (6)}$$

Gauge Transformations

In magnetostatics, it was best to choose $\nabla \cdot A = 0$

In Electrodynamics $\rightarrow$ convenient gauge $\rightarrow$ depends on the problem at hand

Coulomb Gauge and Lorentz Gauge

The Coulomb Gauge

As in magnetostatics, we pick

$$\nabla \cdot A = 0 \quad \text{--- (7)}$$

eqⁿ (4) becomes $\nabla^2 V = -\frac{1}{\epsilon_0}\rho$ --- (8)

Poisson's eqⁿ. We already know how to solve it:

Setting $V=0$ at infinity,

$$V(r,t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r',t)}{\eta}\, d\tau' \quad \text{--- (9)}$$

Unlike electrostatics, V by itself does not tell you E; you have to know A as well (eqⁿ (3))

Scalar potential → in Coulomb gauge
determined by the distribution of charge right now

The advantage of the Coulomb gauge → scalar potential is particularly simple to calculate.

disadvantage → A is particularly difficult to calculate.

eqⁿ (5) in Coulomb gauge reads

$$\nabla^2 A - \mu_0\epsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 J + \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) \quad \text{--- (10)}$$

The Lorentz gauge

In the Lorentz gauge, we pick

$$\boxed{\nabla \cdot A = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}} \quad \text{--- (11)}$$

designed to eliminate the middle term in eq^n (5)

$$\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 J \quad \text{---(12)}$$

eq^n (4) becomes

$$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0} \rho \quad \text{---(13)}$$

Lorentz gauge ⟶ treats V and A on an equal footing

$$\boxed{\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \equiv \Box^2} \quad \text{---(14)}$$

↓

called the d'Alembertian occurs in both equations:

$$\boxed{\begin{aligned} &(i)\ \Box^2 V = -\frac{1}{\varepsilon_0}\rho \\ &(ii)\ \Box^2 A = -\mu_0 J \end{aligned}} \quad \text{---(15)}$$

↓

nice in the context of special relativity, where the d'Alembertian ⟶ natural generalization of the Laplacian and eq^n (15) ⟶ four-dimensional versions of Poisson's eq^ns

In the Lorentz gauge V and A satisfy the inhomogeneous wave equation with a source term on the right.

Electromagnetic Waves in Free Space

Wave? ⟶ means of transporting energy or information

↓

A disturbance of a continuous medium that propagates with a fixed shape at constant velocity.

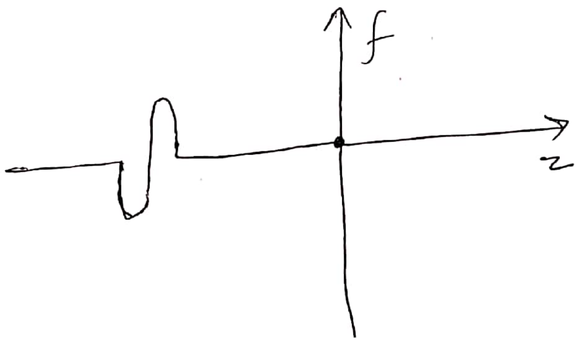
In the presence of absorption?

If the medium is dispersive? these are refinements

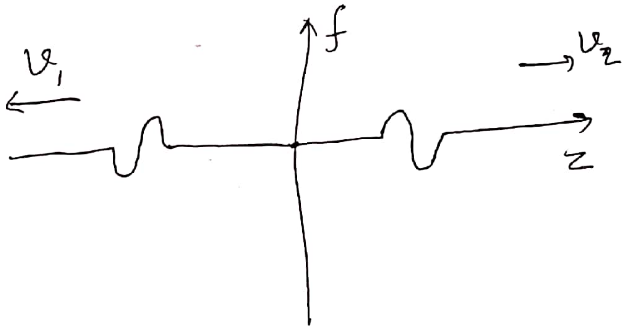
in two or three dimensions?

standing wave?

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \rightarrow \text{Classical wave eq}^n.$$



(a) Incident pulse



(b) Reflected and Transmitted pulse

for a sinusoidal incident wave, the net disturbance of the string is

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z < 0 \\ \tilde{A}_T e^{i(k_2 z - \omega t)} & \text{for } z > 0 \end{cases}$$

Mathematically $f(z,t)$ is continuous at $z=0$

$$f(0^-, t) = f(0^+, t)$$

if the knot itself is of negligible mass

$$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+}$$

Sound waves $\rightarrow$ longitudinal

E.M. Waves $\rightarrow$ Transverse

Electromagnetic Waves

1. free space ($\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$)
2. Lossless dielectrics ($\sigma = 0$, $\varepsilon = \varepsilon_r \varepsilon_0$, $\mu = \mu_r \mu_0$ or $\sigma << \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0$, $\varepsilon = \varepsilon_r \varepsilon_0$, $\mu = \mu_r \mu_0$)
4. Good conductors ($\sigma \simeq \infty$, $\varepsilon = \varepsilon_0$, $\mu = \mu_r \mu_0$ or $\sigma >> \omega\varepsilon$)

Electromagnetic Waves in Vacuum (free Space)

The wave Equations for E and B

region of space $\longrightarrow$ no charge or current

Maxwell's eq^n are

$$\left.\begin{array}{ll} (i)\ \nabla \cdot E = 0 & (iii)\ \nabla \times E = -\dfrac{\partial B}{\partial t} \\ (ii)\ \nabla \cdot B = 0 & (iv)\ \nabla \times B = \mu_0 \varepsilon_0 \dfrac{\partial E}{\partial t} \end{array}\right] \quad (1)$$

Coupled, first order, partial differential eq^s for E and B.

Applying the curl to (iii) and (iv)

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$$

$$= \nabla \times \left(-\frac{\partial B}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B$$

$$= \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times E) = -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Since $\nabla \cdot E = 0$ and $\nabla \cdot B = 0$

$$\boxed{\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \quad , \quad \nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}} \quad \text{--- (2)}$$

Separate eq^ns for E and B $\longrightarrow$ but they are of Second order $\longrightarrow$ price of decoupling ?

In vacuum, each Cartesian component of E and B satisfies the three-dimensional wave equation

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$\Rightarrow$ Maxwell's eq^ns imply that empty space supports the propagation of e.m. waves, travelling at a speed

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s} \quad \text{--- (3)}$$

Crucial role played by Maxwell's contribution to Ampere's Law $\left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$; without it the wave eq^n would not emerge, and there would be no electromagnetic theory of light. !

Monochromatic Plane Waves

Different frequencies in the visible range correspond to different colors, $\longrightarrow$ monochromatic waves

Waves $\longrightarrow$ traveling in the z-direction

$\hookrightarrow$ no x or y dependence

$\downarrow$

We are interested, in fields of the form

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}, \quad \tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)} \quad \text{---(4)}$$

where $\tilde{E}_0$ and $\tilde{B}_0$ $\rightarrow$ (complex) amplitudes

Wave eq^ns (eq^n 2) for E and B were derived from Maxwell's eq^ns. Every solution to Maxwell's eq^ns (in empty space) must obey the wave equation, converse is not true.

Maxwell's eq^ns $\longrightarrow$ impose extra constraints on $\tilde{E}_0$ and $\tilde{B}_0$.

In particular, since $\nabla \cdot E = 0$ and $\nabla \cdot B = 0$, it follows that

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0 \quad \text{---(5)}$$

i.e. $\rightarrow$ electromagnetic waves are transverse: The electric and magnetic fields are perpendicular to the direction of propagation.

From Faraday's Law, $\nabla \times E = -\frac{\partial B}{\partial t}$,

$$\Rightarrow \quad -k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x, \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y \quad \text{---(6)}$$

more compactly

$$\tilde{B}_0 = \frac{k}{\omega}(\hat{z} \times \tilde{E}_0) \quad \text{---(7)}$$

Evidently, E and B are in phase and mutually perpendicular, their amplitudes (real) are related by

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0 \quad \text{---(8)}$$

Maxwell's fourth eq^n $\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$, simply reduces to eq^n (6).

Generalize to monochromatic plane waves travelling in an arbitrary direction.

→ introduction of the propagation (or wave) ~~vector~~ vector, $\vec{k}$, pointing in the direction of propagation.

$$\boxed{\begin{aligned} \tilde{E}(r,t) &= \tilde{E}_0 e^{i(k\cdot r-\omega t)}\,\hat{n} \\ \tilde{B}(r,t) &= \frac{1}{c}\tilde{E}_0 e^{i(k\cdot r-\omega t)}(\hat{k}\times\hat{n}) = \frac{1}{c}\hat{k}\times\hat{E} \end{aligned}} \quad \text{— (9)}$$

$\hat{n}$ → polarization vector. Because E is transverse

$$\hat{n}\cdot\hat{k} = 0 \quad \text{— (10)}$$

Energy and Momentum in Electromagnetic Waves

The energy per unit volume stored in E-M. field is

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \quad \text{— (10)}$$

In case of a monochromatic plane wave

$$B^2 = \frac{1}{c^2}E^2 = \mu_0\varepsilon_0 E^2, \quad \text{— (11)}$$

Electric and magnetic contributions are equal

$$u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2(kz-\omega t+\delta). \quad \text{— (12)}$$

As the wave travels, it carries this energy along with it.

The energy flux density (energy per unit area, per unit time) transported → Poynting vector

$$S = \frac{1}{\mu_0}(E\times B) \quad \text{— (13)}$$

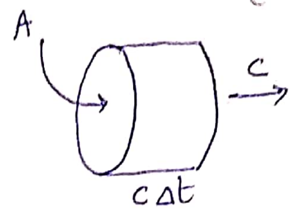
monochromatic waves propagating in the z-direction

$$S = c\varepsilon_0 E_0^2 \cos^2(kz-\omega t+\delta)\,\hat{z} = cu\hat{z}$$

— (14)

$\underline{S}$ → energy density (u) times the velocity of the waves ($c\hat{z}$)

In a time Δt, a length $c\Delta t$ passes through area A, carrying energy $\underline{uAc\Delta t}$ with it.



Energy per unit time, per unit time, transported by the wave is → $\underline{uc}$

Electromagnetic fields → carry energy as well as $\underline{\text{momentum}}$

Momentum density stored in the field is

$$\mathcal{P} = \frac{1}{c^2} S \quad \text{—(15)}$$

for monochromatic plane waves, then

$$\mathcal{P} = \frac{1}{c}\varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{z} = \frac{1}{c}u\hat{z} \quad \text{—(16)}$$

$$\langle u \rangle = \frac{1}{2}\varepsilon_0 E^2 \quad \text{—(17)}$$

$$\langle S \rangle = \frac{1}{2}c\,\varepsilon_0 E_0^2\hat{z} \quad \text{—(18)}$$

$$\langle \mathcal{P} \rangle = \frac{1}{2c}\varepsilon_0 E_0^2\hat{z} \quad \text{—(19)}$$

* Average of cosine-squared over a complete cycle is $\frac{1}{2}$

Average power per unit area transported by an E.M. wave → Intensity

$$I \equiv \langle S \rangle = \frac{1}{2}c\varepsilon_0 E^2 \quad \text{—(20)}$$

When light falls on a perfect absorber → it delivers its momentum to the surface. In time Δt the momentum transfer is $\Delta p = \langle \mathcal{P} \rangle Ac\Delta t$

⇒ radiation pressure $$P = \frac{1}{A}\frac{\Delta p}{\Delta t}$$

$$= \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c} \quad \text{—(21)}$$